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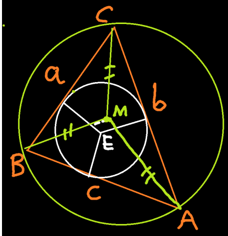
Some basic tools in Geometry

Geometry

- Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
- cosine Law: → $a^2 = b^2 + c^2 - 2bc \cdot \cos A$



→ if $\hat{A} = 90^\circ \rightsquigarrow a^2 = b^2 + c^2$ (Pyth. Thm) $\Longleftrightarrow$

→ if $\hat{A} < 90^\circ \rightsquigarrow a^2 < b^2 + c^2$ $\Longleftrightarrow$

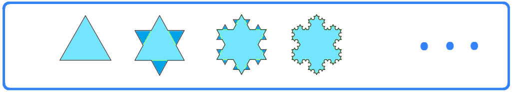
→ if $\hat{A} > 90^\circ \rightsquigarrow a^2 > b^2 + c^2$ $\Longleftrightarrow$

1) angular bisectors ⇝ incentre E.
2) perp. bisectors ⇝ circumcentre M.
3) medians ⇝ barycentre (centre of gravity)
4) altitudes ⇝ (orthocentre)

♣ **Today:** lengths, angles, perimeters, areas

Can the perimeter of a shape be infinite?

There are shapes which have a finite area but infinite perimeter!
Such an example is the Koch snowflake, which is a fractal that can be built up iteratively, in a sequence of stages. At the first stage we consider an equilateral triangle, and then recursively altering each line segment as follows:



step 1) divide the line segment into three segments of equal length.
step 2) draw an equilateral triangle that has the middle segment from step 1 as its base and points outward.
step 3) remove the line segment that is the base of the triangle from step 2.
The Koch snowflake is the outcome if we keep repeating this process forever! Although the Koch snowflake is a continuous curve, drawing a tangent line to any point is impossible! It is not difficult to check that the perimeters of the successive stages increase without bound, which makes the perimeter of the Koch snowflake infinite. On the other hand, the Koch snowflake can be inscribed in a disc, and so it has finite area!

If we know the lengths of the 3 sides of a triangle and have no other information what conclusions can we make?

1) We can check if indeed these lengths correspond to the 3 sides of a triangle (which is not always the case)!

The lengths a, b, c correspond to the 3 sides of a triangle if and only if they satisfy the following inequalities: $a < b + c$, $b < a + c$, $c < a + b$.

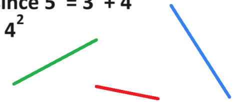
For example, we cannot form a triangle of sides with lengths 3, 2, 1 because we must have $3<2+1$ (which is not true), but we can form a triangle of sides with lengths 3, 4, 6 since $3<4+6$, $4<3+6$, $6<3+4$.

2) We can detect which type of angles we have (and in fact compute them)!

For example, if the lengths are 3, 4, 5 we have a right triangle since $5^2 = 3^2 + 4^2$ and if they are 3, 4, 6 we have an obtuse triangle since $6^2 > 3^2 + 4^2$

3) We can compute the area of the triangle!

To do this we will prove Heron's formula.



Problems

Problem 1.

Consider a triangle whose sides have lengths a, b, and c and let s be its semiperimeter; that is,

$$s = \frac{a+b+c}{2}.$$

Show that its area is given by: $A = \sqrt{s(s-a)(s-b)(s-c)}$.

The above formula is known as Heron's formula.

♣ Question: If two triangles have the same perimeter, do they have the same area?

♦ Answer: No! Consider a right angle triangle of sides 3, 4 and 5 and an equilateral triangle of side 4.

Problem 1: Solution

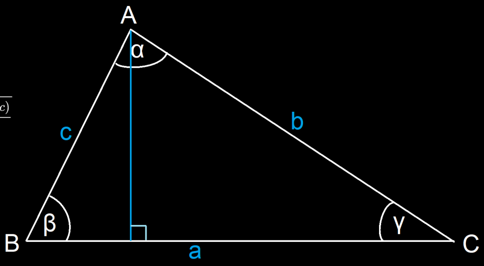
The altitude of the triangle on base a has length $b\sin\gamma$, and it follows

$$A = \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}ab\sin\gamma.$$

Applying the law of cosines we get $\cos\gamma = \dfrac{a^2+b^2-c^2}{2ab}$ and so $\sin\gamma = \sqrt{1-\cos^2\gamma} = \dfrac{\sqrt{4a^2b^2-(a^2+b^2-c^2)^2}}{2ab}$.

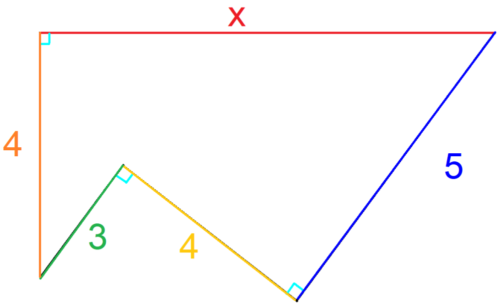
It follows

$$\begin{aligned}
A &= \frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \\
&= \frac{1}{4}\sqrt{(2ab-(a^2+b^2-c^2))(2ab+(a^2+b^2-c^2))} \\
&= \frac{1}{4}\sqrt{(c^2-(a-b)^2)((a+b)^2-c^2)} \\
&= \sqrt{\frac{(c-(a-b))(c+(a-b))((a+b)-c)((a+b)+c)}{16}} \\
&= \sqrt{\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}\frac{(a+b+c)}{2}} \\
&= \sqrt{\frac{(a+b+c)}{2}\frac{(b+c-a)}{2}\frac{(a+c-b)}{2}\frac{(a+b-c)}{2}} \\
&= \sqrt{s(s-a)(s-b)(s-c)}.
\end{aligned}$$



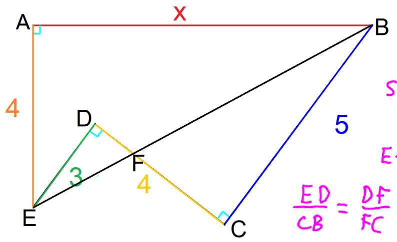
Problems

Problem 2.



Compute x.

♦ Problem 2: Solution



Bring EB and consider the point F of the intersection of EB with DC.

Let $a = DF$, $b = EF$, $c = FB$.

Since $E\hat{F}D = C\hat{F}B$ and $E\hat{D}F = F\hat{C}B = 90^\circ$, we have that $E\hat{F}D \approx C\hat{F}B$ (similar) and so:

$$\frac{ED}{CB} = \frac{DF}{FC} = \frac{EF}{BF} \Leftrightarrow \frac{3}{5} \overset{①}{=} \frac{a}{4-a} \overset{②}{=} \frac{b}{c}.$$

From ①: $12 - 3a = 5a \Leftrightarrow a = 12/8 = \boxed{3/2}$

From Pythagoras' Thm: $EF^2 = ED^2 + DF^2 \Leftrightarrow b = \sqrt{3^2 + \frac{3^2}{2^2}} = \boxed{\frac{3}{2}\sqrt{5}}$

From ② (or Pythagoras' Thm): $c = \boxed{\frac{5}{2}\sqrt{5}}$; hence

$EB = b + c = \boxed{4\sqrt{5}}$

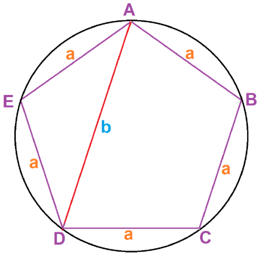
From Pythagoras' Thm:

$x = \sqrt{EB^2 - AE^2} = \sqrt{(4\sqrt{5})^2 - 4^2} = \sqrt{4^2(5-1)} = 8$

Problems

Problem 3.

Given a regular pentagon which has side length a and (common) distance b of any two non-adjacent vertices, show that $\dfrac{b}{a} = \dfrac{1+\sqrt{5}}{2} =: \phi$ (golden ratio).

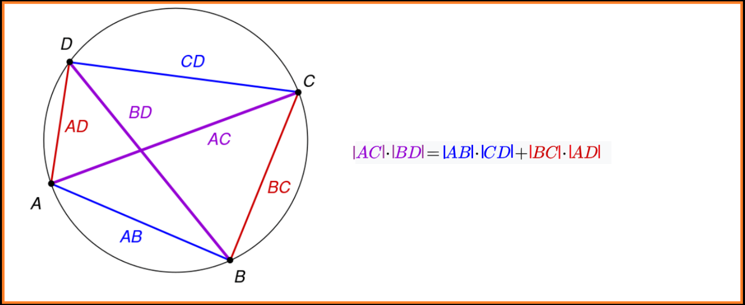


♦ Problem 3: Solution

To solve this problem we will use the following important result:

Ptolemy's theorem

If a quadrilateral is inscribable in a circle then the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of opposite sides.



If the vertices of the *inscribable* quadrilateral are A, B, C, and D in order, then

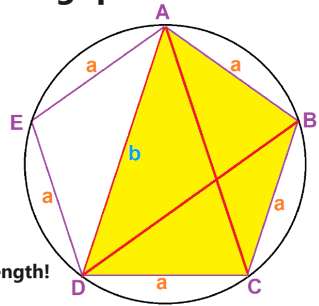
$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|$$

♣ **Note:** The converse is also true! If the conclusion of Ptolemy's theorem is satisfied then the quadrilateral is cyclic!

♦ Problem 3: Solution

We consider the cyclic quadrilateral ABCD and bring its two diagonals. By Ptolemy's Theorem we get: $|AC||BD|=|AB||CD|+|BC||DA|$, or equivalently, $bb = aa + ab$. Treating b as our variable and a as a constant, we have to solve the following quadratic equation:

$$b^2 - ab - a^2 = 0$$



The solutions are:

$$b = \frac{a \pm \sqrt{a^2 + 4a^2}}{2} = \frac{(1 \pm \sqrt{5})a}{2}$$

but we accept only the positive solution since it represents length!

Problems

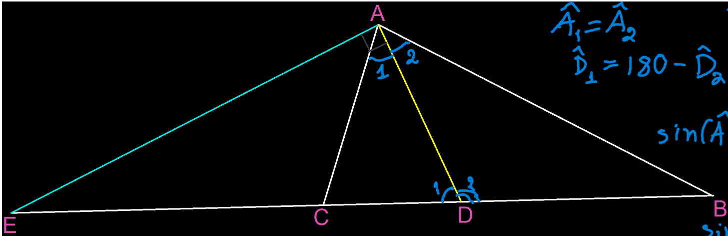
Problem 4.
(Interior/Exterior Angle Bisector Theorem) Consider a triangle ABC and the bisector AD of the angle $\angle A$. Show that

$$\frac{DB}{DC} = \frac{AB}{AC}.$$

Moreover, if AE is the external bisector of the angle $\angle A$, show that

$$\frac{EB}{EC} = \frac{AB}{AC}.$$

♦ Problem 4: Solution



$$\left.\begin{aligned}\hat{A}_1 &= \hat{A}_2\\ \hat{D}_1 &= 180 - \hat{D}_2\end{aligned}\right\} \Longrightarrow$$

$\sin(\hat{A}_1) = \sin(\hat{A}_2)$ and $\sin(\hat{D}_1) = \sin(\hat{D}_2)$

The sine law for $A\overset{\triangle}{C}D$ and $A\overset{\triangle}{D}B$ implies:

$$\frac{AC}{DC} = \frac{\sin(\hat{D}_1)}{\sin(\hat{A}_1)} \quad \text{and} \quad \frac{AB}{DB} = \frac{\sin(\hat{D}_2)}{\sin(\hat{A}_2)} \Longrightarrow \frac{AC}{DC} = \frac{AB}{DB} \Longleftrightarrow$$

$$\frac{DB}{DC} = \frac{AB}{AC}.$$

Similarly, the sine law for $E\overset{\triangle}{C}A$, $A\overset{\triangle}{E}B$ gives

$$\frac{EB}{EC} = \frac{AB}{AC}.$$

Problems

Problem 5.
Let A and B be two fixed points in the plane. Find the geometric locus of points in the plane such that their distances from A and B have a constant (and known) ratio $\frac{m}{n}$.

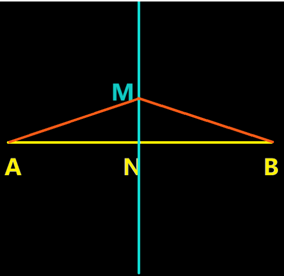
♠ Hint: separate two cases: i) $\frac{m}{n} = 1$ and ii) $\frac{m}{n} \neq 1$.

The circles that occur in the second case are called Apollonian circles.

♣ Question: What is the geometric locus that occurs if we replace the constant ratio by constant sum?

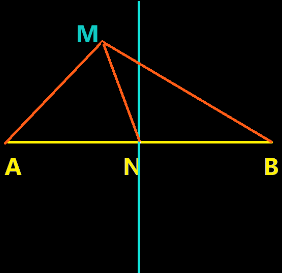
♦ Answer: An ellipse!

♦ Problem 5: Solution (case i)



Claim: if the ratio is 1, then the locus is the perpendicular bisector of the segment AB.

Indeed, if M is any point on the perpendicular bisector of the segment AB, and N is the midpoint of AB, then the triangles AMN and BMN are equal since the angles ANM and BNM are equal (as right angles), AN=NB and MN is common. Thus AM=BM, i.e. the ratio is 1.



Conversely, if M is a point such that AM/BM=1, then the triangles AMN and BMN are equal since AM=BM, AN=NB and MN is common. This implies that the angles ANM and BNM are equal, and so they are right angles (since their sum is 180^o).
Hence, M lies on the perpendicular bisector of the segment AB.

♦ Problem 5: Solution (case ii)

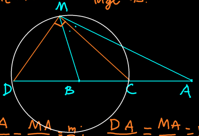
Solutions : Problem 5 : When we need to find a geom. locus.

1) guess the locus (experiments) (ex. curve on which my points lie)

2) check that each point on the candidate curve, belongs to the locus

1) Let M be on the locus: $\frac{MA}{MB} = \frac{m}{n}$

MD: exterior angle bis. of $B\hat{M}A$
MC: interior angle bis.



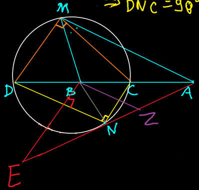
$\frac{CA}{CB} = \frac{MA}{MB} = \frac{m}{n}$, $\frac{DA}{DB} = \frac{MA}{MB} = \frac{m}{n}$

⇒ D and C are also the locus

⇒ M belongs to the unique circle with diameter DC

guess which is the locus

2) Conversely, if N belongs to the circle we will show that $\frac{NA}{NB} = \frac{m}{n}$ (i.e. that N belongs to the locus)



⇒ $D\hat{N}C = 90°$

Bring BE // NC

ABE : $\frac{NA}{NE} = \frac{CA}{CB} = \frac{m}{n}$ (Thales Thm)

Bring BZ // DN :

ADN : $\frac{NA}{NZ} = \frac{AD}{DB} = \frac{m}{n}$ (Thales thm)

⇒ NZ = NE : i.e. N: median of EZ

$E\hat{B}Z = 90°$

⇒ BN = NE = NZ

$\frac{NA}{NB} = \frac{NA}{NZ} = \frac{m}{n}$

☺ ⇒ N belongs to the locus

verify that all points on the "candidate locus" indeed belong to the locus.

Problems (homework)

Problem 6. Let Q be a point inside a triangle ABC. Three lines pass through Q and are parallel with the sides of the triangle. These lines divide the initial triangle into six parts, three of which are triangles of areas S_1, S_2 and S_3. Prove that

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

⧫ Problem 6: Solution

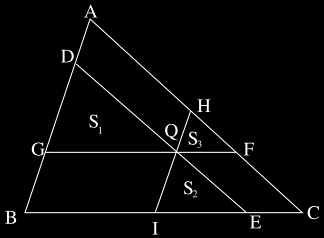
Let D, E, F, G, H, I be the points of intersection between the three lines and the sides of the triangle.

Then triangles DGQ, HQF, QIE and ABC are similar so

$$\frac{S_1}{[ABC]} = \left(\frac{GQ}{BC}\right)^2 = \left(\frac{BI}{BC}\right)^2$$

Similarly

$$\frac{S_2}{[ABC]} = \left(\frac{IE}{BC}\right)^2, \quad \frac{S_3}{[ABC]} = \left(\frac{QF}{BC}\right)^2 = \left(\frac{CE}{BC}\right)^2.$$



Then

$$\sqrt{\frac{S_1}{[ABC]}} + \sqrt{\frac{S_2}{[ABC]}} + \sqrt{\frac{S_3}{[ABC]}} = \frac{BI}{BC} + \frac{IE}{BC} + \frac{EC}{BC} = 1.$$

This yields

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

Problems (homework)

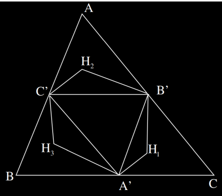
Problem 7. Let $A'B'C'$ be the median triangle of ABC and denote by H_1, H_2 and H_3 the orthocenters of triangles $CA'B'$, $AB'C'$ and $BC'A'$ respectively.

Prove that:

(i) $[A'H_1B'H_2C'H_3] = \frac{1}{2}[ABC]$.

(ii) If we extend the line segments AH_2, BH_3 and CH_1, then they will all 3 meet at a point.

Problem 7: Solution

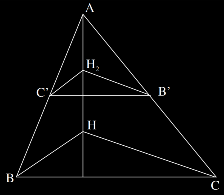


(i) First remark that $A'B'C'$ and ABC are similar triangles with the similarity ratio $B'C' : BC = 1 : 2$. Therefore

$$[A'B'C'] = \frac{1}{4}[ABC].$$

Let H be the orthocenter of ABC. Then A, H_2 and H are on the same line. Also triangles $H_2C'B'$ and HBC are similar with the same similarity ratio, thus

$$[H_2B'C'] = \frac{1}{4}[HBC].$$



In the same way we obtain

$$[H_1A'B'] = \frac{1}{4}[HAB] \quad \text{and} \quad [H_3C'A'] = \frac{1}{4}[HCA].$$

We now obtain

$$\begin{aligned}[A'H_1B'H_2C'H_3] &= [A'B'C'] + [H_1A'B'] + [H_2B'C'] + [H_3C'A'] \\ &= \frac{1}{4}[ABC] + \frac{[HAB]+[HBC]+[HCA]}{4} \\ &= \frac{1}{4}[ABC] + \frac{1}{4}[ABC] = \frac{1}{2}[ABC].\end{aligned}$$

(ii)Remark that the extensions of AH_2, BH_3 and CH_1 are the altitudes of the triangle ABC. Hence they all meet at a point (namely the orthocenter of ABC).

Problems (homework)

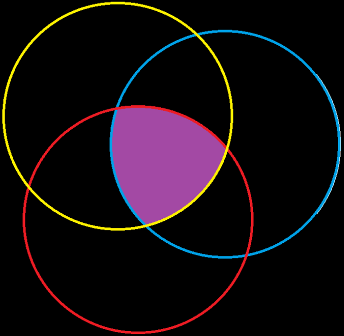
Problem 8.

(i) Find a shape (with no holes) that has constant width, but is **not** a circle.

(ii) For the shape you found in part (i), compute its perimeter as a function of its (constant) width w. What do you observe?

(iii) For the shape you found in part (i), compute its area as a function of its (constant) width w.

♦ Problem 8: Solution

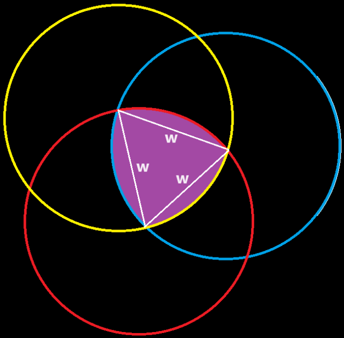


We observe that the Reuleaux triangle has the same perimeter with the circle of the same width! This is not a coincidence: Barbier's Theorem tells us that all shapes of constant width w have perimeter πw!

(i) Consider the purple shape, which the intersection of 3 discs of equal radii which have the following property: the centre of each disc coincides with one (of the 2) intersection points of the other 2 discs. It is easy to see that the width of each shape is constant (in fact it is equal to the radius of the discs).

(ii) The 3 vertices of the purple shape (called Reuleaux triangle) form an equilateral triangle. Thus, each side is an arc of 60°, and so its length is 60/360 times the perimeter of each circle which is $2\pi w$. Hence the perimeter of the Reuleaux triangle is πw.

⧫ Problem 8: Solution



(iii) Since the area of each meniscus-shaped portion of the Reuleaux triangle is a circular arc with opening angle 60 , we get that its area (let's call it M) equals the area of the cyclic sector minus the area of the equilateral triangle. Thus, we have

$$M = (60/360)\pi w^2 - (\sqrt{3}/4)w^2$$

Hence the total area A of the Reuleaux triangle is equal to 3M minus the area of the equilateral triangle:

$$A = 3M - (\sqrt{3}/4)w^2 = \frac{(\pi - \sqrt{3})w^2}{2}$$

In fact, the Reuleaux triangle has the smallest area for a given width of any curve of constant width w!

Some useful/interesting links:

- https://imogeometry.blogspot.com/p/1.html
- http://www.imo-official.org/problems.aspx
- https://www.geogebra.org/t/geometry
- https://thatsmaths.com/tag/geometry/
- https://www.ucd.ie/mathstat/newsandevents/events/mathsenrichment/
- https://en.wikipedia.org/wiki/Apollonian_circles
- https://en.wikipedia.org/wiki/Reuleaux_triangle
- https://www.youtube.com/watch?v=cUCSSJwO3GU&t=287s&ab_channel=Numberphile